Fluid and Mean Field Approximations: fluid approximation in continuous time

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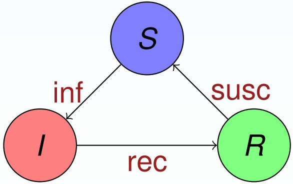
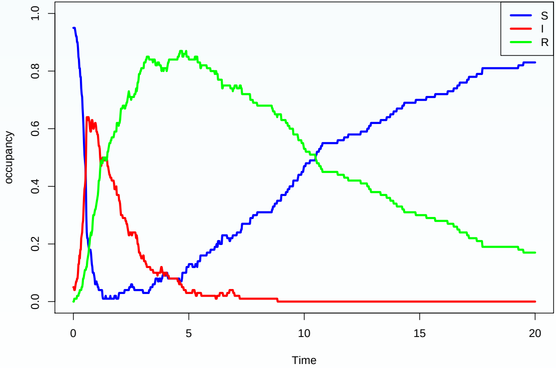
EXAMPLE: SIR EPIDEMICS

Population model
$N = 100$ agents





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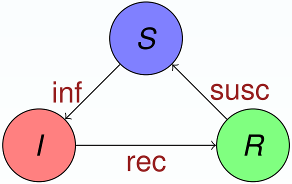

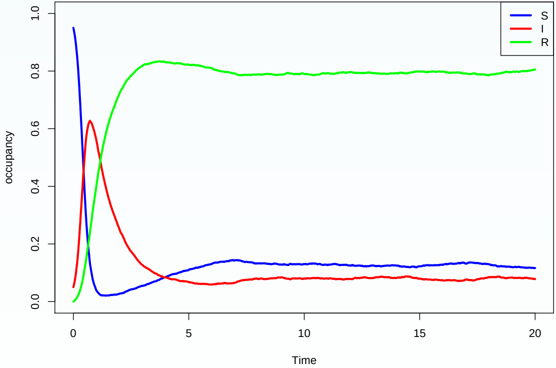

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OVERVIEW

We will consider Markov models of population processes: systems composed of populations of interacting agents, whose behaviour is a collective emergent property.

MEAN FIELD/ FLUID APPROXIMATION

Approximation by a deterministic system (differential/ difference equations).

MEAN FIELD (ORIGINALLY)/ FAST SIMULATION

Approximation by another, simpler, stochastic model.

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OVERVIEW: FLUID APPROXIMATION

LIMIT THEOREM POINT OF VIEW

Considers the deterministic model as the limit of the stochastic process for large populations/ system size:

- CTMC to ODE
- DTMC to Difference Equations
- DTMC to ODE
- CTMC to Gaussian processes (central limit)
- CTMC to hybrid system
- CTMC to SDE (diffusion limit)

MOMENT CLOSURE POINT OF VIEW

Considers the deterministic model as an approximation of the mean of the stochastic process.
Equations for higher order moments can be given as well.

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Overview: mean field

Approximation by another, simpler, stochastic model.

Fast simulation

Approximate the behaviour of one or few agents by another stochastic process depending on the mean of the rest of the system.

Hartree approximation (mean field)

Approximates the process (at transient/ steady state) by assuming a product form (w.r.t. variables). The decoupling is obtained by averaging the rates of transitions acting on a variable X with respect to the other variables.

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MENU À LA CARTE

- Fluid approximation (CTMC + ODE)
- Mean Field (DTMC + discrete time limit)
- Steady state limits
- Error bounds
- Fluid equation and moments, system-size expansion
- Central Limit and linear noise approximation
- Product form approximation (Hartree, variational)
- Hybrid mean field
- Fluid model checking

Outline

Outline

1. Continuous-Time Markov Chains: a primer
   - Poisson Process
2. Population CTMC
3. Fluid Approximation
4. Infinitesimal Generators
5. Steady State approximation
6. Rewards

EXPONENTIAL DISTRIBUTION

DEFINITION

A random variable $T : (\Omega, \mathcal{S}) \rightarrow [0, \infty]$ is $Exp(\lambda)$ iff

- Cdf is $\mathbb{P}(T < t) = 1 - e^{-\lambda t}$
- Density is $f_T(t) = \lambda e^{-\lambda t}$, $t \geq 0$.

The expected value of T is $\mathbb{E}(T) = \int_0^\infty \mathbb{P}(T > t)dt = \frac{1}{\lambda}$.

MEMORYLESS PROPERTY

$T \sim Exp(\lambda)$ if and only if the following memoryless property holds:

$$\mathbb{P}(T > s + t | T > s) = \mathbb{P}(T > t) \text{ for all } s, t \geq 0.$$

INSTANTANEOUS FIRING PROBABILITY

An exponential distribution with rate λ can be seen as the firing time of an event who has probability of firing between time t and $t + dt$ equal to λdt.

CTMC: DEFINITION

S-VALUED CONTINUOUS TIME MARKOV CHAIN

- Let S be finite or countable.
- A CTMC on a state space S is a labelled graph, where labels are the rates of exponential distributions.
- In each state, there is a race condition between the different exiting edges: the fastest is traversed.
- The CTMC has the memoryless property: the future depends only on the current state.

FORMALLY

A Continuous Time Markov Chain is a right-continuous continuous-time random process (with cadlag sampling paths) satisfying the memoryless condition: for each n, t_i and s_i:

$$\mathbb{P}(X_{t_n} = s_n \mid X_{t_0} = s_0, \ldots, X_{t_{n-1}} = s_{n-1}) = \mathbb{P}(X_{t_n} = s_n \mid X_{t_{n-1}} = s_{n-1}).$$

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CTMC: INFINITESIMAL GENERATOR

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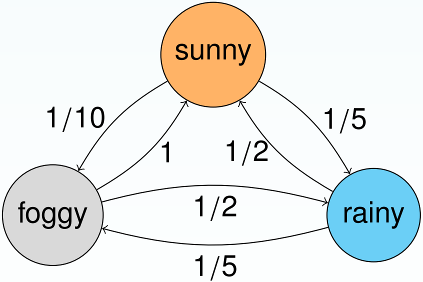
Q-MATRIX

A Q-matrix is the $|S| \times |S|$ matrix such that:

1. $q_{ij} \geq 0$, $i \neq j$ is the rate of the exponential distribution giving the time needed to go from state s_i to state s_j
2. $q_{ii} = -\sum_{j \neq i} q_{ij}$ is the opposite of the exit rate from state i.

Therefore, each row of the Q-matrix sums up to zero.

A SIMPLE EXAMPLE: WEATHER CHAIN



$$S = \{sunny, rainy, foggy\}$$

$$Q = \begin{pmatrix} -\frac{3}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{2} & -\frac{7}{10} & \frac{1}{5} \\ 1 & \frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

JUMP CHAIN AND HOLDING TIMES

FACTORIZING EACH JUMP

In each state i, we have a race condition between k transitions, each exponentially distributed with rate q_{ij}. Hence, the time spent is $T = \inf T_{ij}$. By the properties of the exponential distribution, we know that T has rate $q_i = \sum_j q_{ij}$, and that the transition that fires is independent from T and the next state j is chosen with probability q_{ij}/q_i.

JUMP CHAIN AND HOLDING TIMES

We can therefore factorize $X(t)$ into

- a DTMC Y_n, with probability matrix Π, defined by $\pi_{ij} = \frac{q_{ij}}{-q_{ii}}$, if $i \neq j$, and $\pi_{ii} = 0$;
- a sequence of jump times τ_n, where τ_n is the time of the n-th jump. Letting q_i the jump rate from state s_i, then $T_n = \tau_n - \tau_{n-1}$, the n-th holding time, is distributed exponentially with rate q_{Y_n}.
- Y_n and each T_i are independent.
- Hence $X(t) = Y_n$ for $\tau_n \leq t < \tau_{n+1}$.

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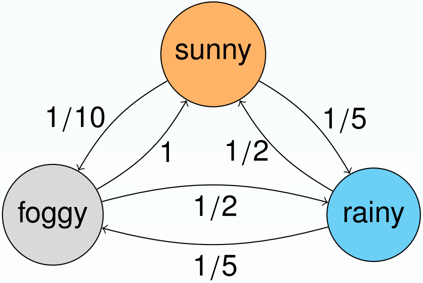

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Jump chain

$$\Pi = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{5}{7} & 0 & \frac{2}{7} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Exit rates

$$q = \left(\frac{3}{10}, \frac{7}{10}, \frac{3}{2}\right)$$

CHAPMAN-KOLMOGOROV EQUATIONS

Let $P_{ij}(t) = \mathbb{P}\{X(t) = s_j \mid X(0) = s_i\}$. Then

$$\begin{aligned} P_{ij}(t+s) &= \mathbb{P}\{X(t+s) = s_j \mid X(0) = s_i\} \\ &= \sum_k \mathbb{P}\{X(t+s) = s_j, X(t) = s_k \mid X(0) = s_i\} \\ &= \sum_k \mathbb{P}\{X(s) = s_j \mid X(0) = s_k\}\mathbb{P}\{X(t) = s_k \mid X(0) = s_i\} \\ &= \sum_k P_{ik}(s)P_{kj}(t). \end{aligned}$$

Hence $P(t)$, as a matrix, satisfies

$$P(t+s) = P(t)P(s) = P(s)P(t),$$

which is the semigroup property, also known as Chapman-Kolmogorov equations.

KOLMOGOROV EQUATIONS

Using properties of the exponential, we can compute $P(dt)$:

- $P_{ij}(dt) = q_{ij}dt$, for $i \neq j$;
- $P_{ii}(dt) = 1 - \sum_{j \neq i} q_{ij}dt = 1 + q_{ii}dt$

Hence $P(dt) = I + Qdt$

From the CK equations: $P(t + dt) = P(t) + P(t)Qdt$, from which

$$\frac{dP(t)}{dt} = P(t)Q,$$

which is the forward Kolmogorov equation.

Using CK the other way round: $P(t + dt) = P(t) + QP(t)dt$, so

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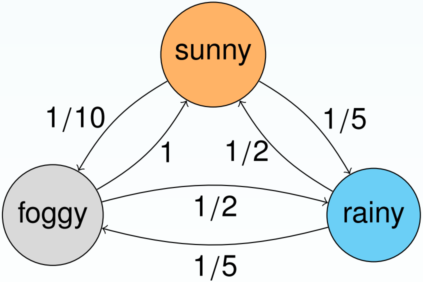

$$S = \{sunny, rainy, foggy\}$$

$$p_0 = (0, 1, 0) \quad p = p_0 P$$

$$\frac{d}{dt} p_0 P = p_0 P Q \Rightarrow \frac{d}{dt} p = pQ$$

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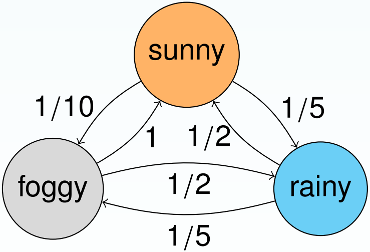
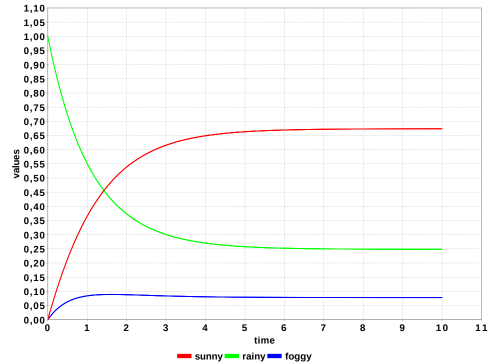
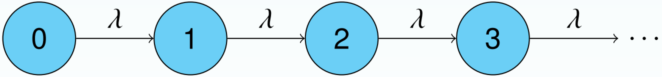




POISSON PROCESS: DEFINITION

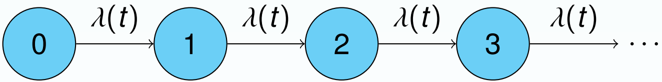
A Poisson process $\mathcal{N}_\lambda(0,t)$ with rate λ is a process that counts how many times an exponential distribution with rate λ has fired from time 0 to time t.



- It can be seen as a CTMC on the state space $S = \mathbb{N}$, with rate matrix Q given by $q_{i,i+1} = \lambda$, and zero elsewhere.
- It's a very common process. For instance, it is the simplest model of job arrivals in a queue.
- It can be shown that the distribution of $\mathcal{N}_\lambda(0,t)$ is *Poisson*(λt)

TIME-INHOMOGENEOUS POISSON PROCESS

A time-inhomogeneous Poisson process $\mathcal{N}_\lambda(0,t)$, $\lambda = \lambda(t)$, is a Poisson process with time-varying rate.



It can be shown that the distribution of $\mathcal{N}_\lambda(0,t)$ is *Poisson*$(\Lambda(t))$, where

$$\Lambda(t) = \int_0^t \lambda(s)ds.$$

Outline

POPULATION CTMC

If we want to describe population processes, with many agents, representing the CTMC by its Q-matrix is unfeasible, as the state space blows up.

A population CTMC model is a tuple $\mathcal{X} = (\mathbf{X}, \mathcal{D}, \mathcal{T}, \mathbf{x_0})$, where:

1. $\mathbf{X}$ — vector of *variables* counting how many individuals in each state.
2. $\mathcal{D} = \prod_i \mathcal{D}_i$ — (countable) state space.
3. $\mathbf{x_0} \in \mathcal{D}$ —*initial state*.
4. $\eta_i \in \mathcal{T}$ — *global transitions*, $\eta_i = (a, \phi(\mathbf{X}), \mathbf{v}, r(\mathbf{X}))$
   1. a — event name (optional).
   2. $\phi(\mathbf{X})$ — guard.
   3. $\mathbf{v} \in \mathbb{R}^n$ — *update vector* (from $\mathbf{X}$ to $\mathbf{X} + \mathbf{v}$)
   4. $r : \mathcal{D} \to \mathbb{R}_{\geq 0}$ — rate function.

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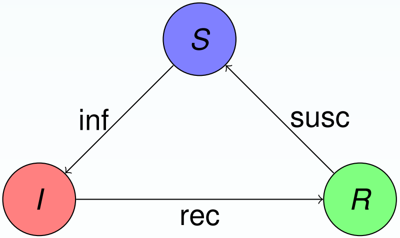

Three variables: X_S,X_I,X_R.
State space:
$\mathcal{D} = \{(n_1, n_2, n_3) \mid n_1 + n_2 + n_3 = N\} \subset \{0, \ldots, N\}^3$.

Transitions:

- $(inf, \top, (-1, 1, 0)k_I \frac{X_I}{N} X_S)$
- $(rec, \top, (0, -1, 1), k_R X_I)$
- $(susc, \top, (1, 0, -1), k_S X_R)$

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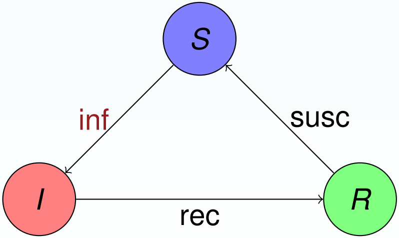

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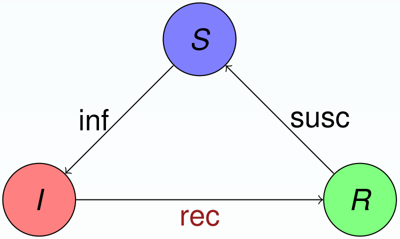

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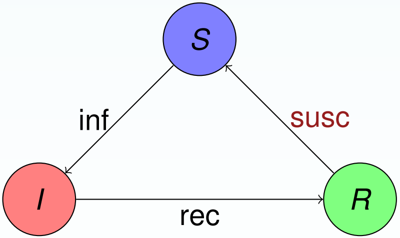

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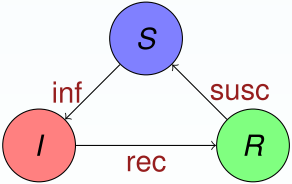
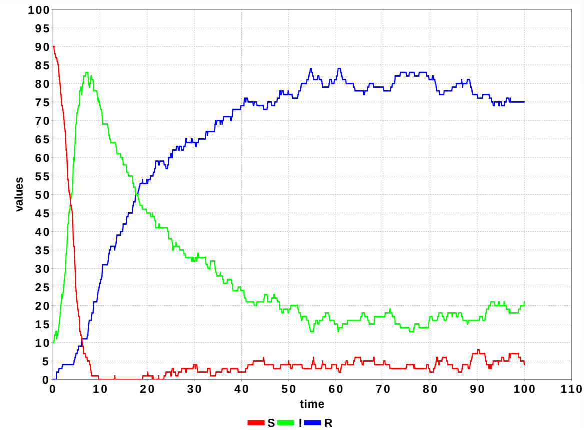


$N = 100,\ k_I = 1,\ k_R = 0.05,\ k_S = 0.01$



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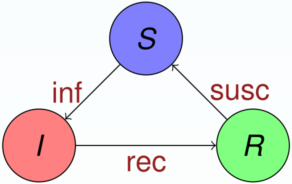

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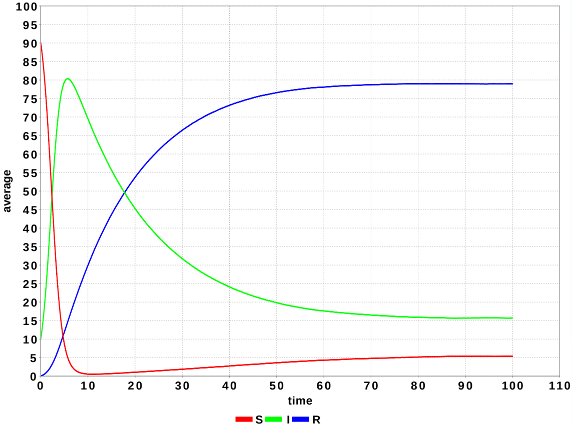

(average)

MASTER EQUATION

The Kolmogorov equation in the context of Population Processes is often know as master equation.

There is one equation per state $\mathbf{x} \in \mathcal{D}$, for the probability mass $P(\mathbf{x}, t)$, which considers the inflow and outflow of probability at time t.

$$\frac{dP(\mathbf{x}, t)}{dt} = \sum_{\eta \in \mathcal{T}} r_\eta(\mathbf{x} - \mathbf{v}_\eta) P(\mathbf{x} - \mathbf{v}_\eta, t) - \sum_{\eta \in \mathcal{T}} r_\eta(\mathbf{x}) P(\mathbf{x}, t)$$

POISSON REPRESENTATION

Population CTMC admit a simple description in terms of Poisson processes (random time change).
Essentially, we introduce variables $R_\eta(t)$ counting how many times each transition η has fired up to time t. Hence we can write:

$$X(t) = X(0) + \sum_{\eta \in \mathcal{T}} \mathbf{v}_\eta R_\eta(t).$$

It turns out that $R_\eta(t)$ is a time-inhomogeneous Poisson process with cumulative rate $\int_0^t r_\eta(X(s))ds$, independent from the other $R_{\eta'}$.
Hence, let $\mathcal{N}_\eta$ be independent Poisson processes. For each $t \geq 0$:

$$X(t) = X(0) + \sum_{\eta \in \mathcal{T}} \mathbf{v}_\eta \mathcal{N}_\eta\left(\int_0^t r_\eta(X(s))ds\right).$$

Outline

FLUID APPROXIMATION

BASICS

- It applies to CTMC models of population dynamics with large population size N (studies the limit as $N \to \infty$)
- It works on scaled variables, to treat uniformly different population levels.
- Requires proper scaling and regularity assumptions on rates.
- The method works by constructing an ODE from the sequence of population dependent CTMC.
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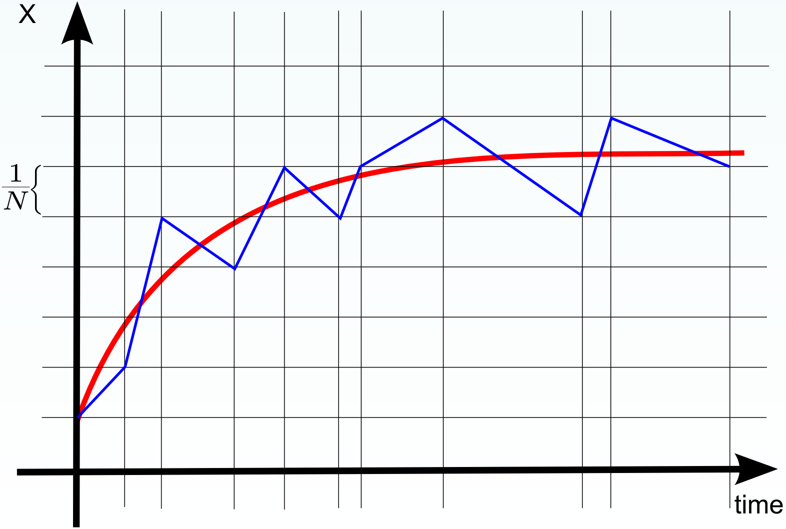
AN INTUITION



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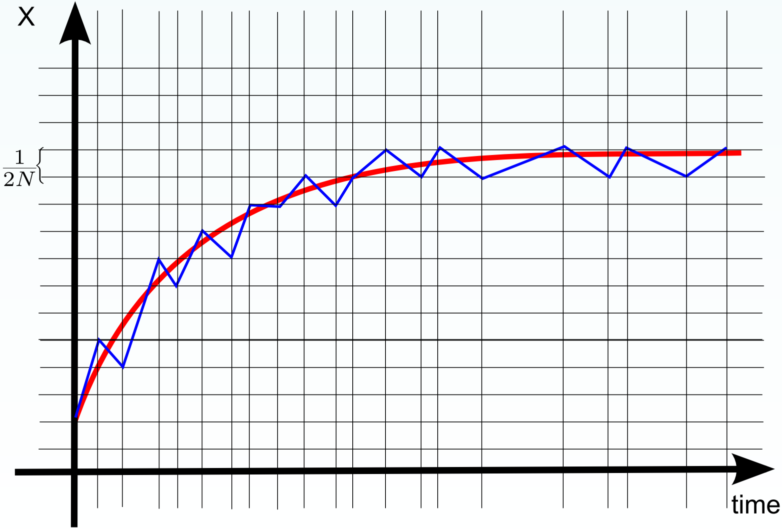
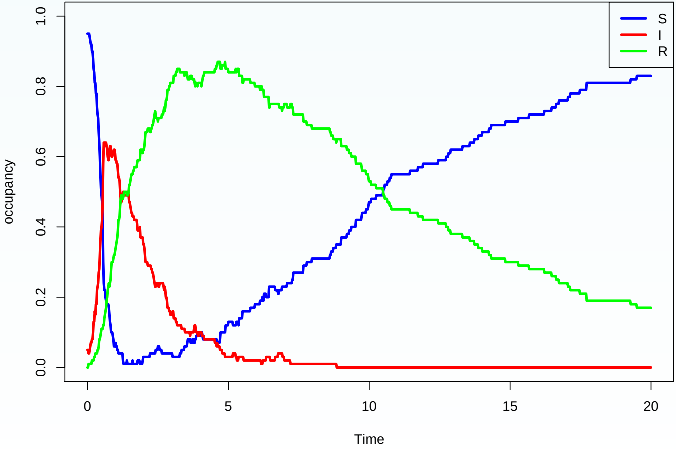
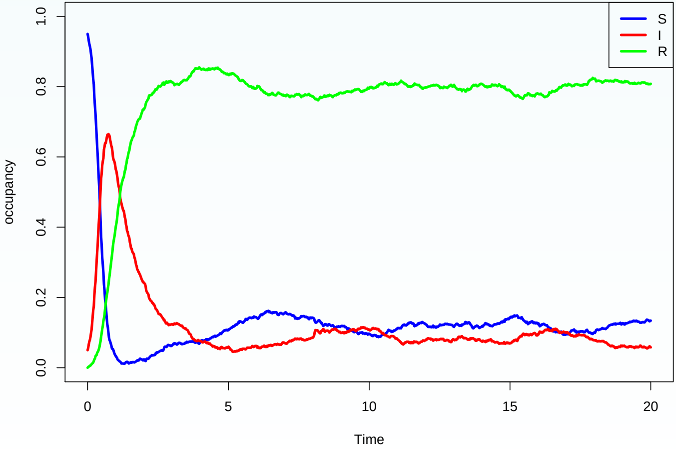


EXAMPLE CONTINUED



CTMC $N = 100$

EXAMPLE CONTINUED



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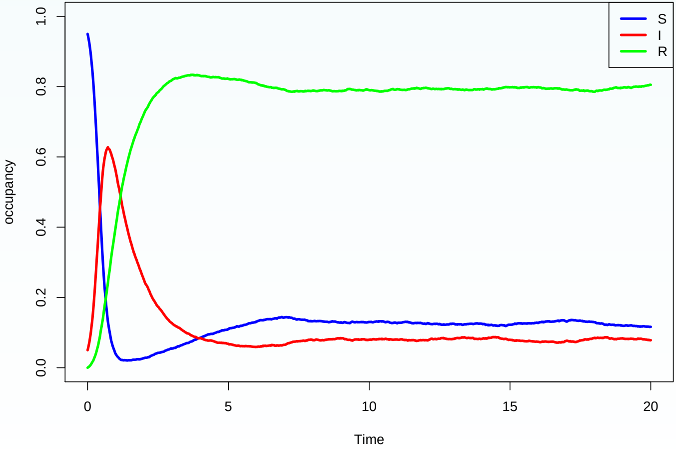

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SCALING CONDITIONS

BASICS

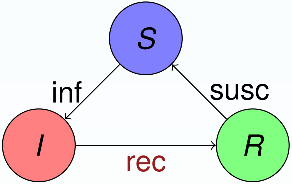
- We have a sequence $\mathcal{X}^{(N)}$ of models, for increasing system size (e.g. total population N).
- We normalize such models in order to bring them to the same scale (divide variables by size N).
- $\mathbf{X}^{(N)}(t)$ is the Markov process (in continuous time) defined by $\mathcal{X}^{(N)}$.

NORMALIZATION

The normalized model $\hat{\mathcal{X}}^{(N)} = (\hat{\mathbf{X}}, \hat{\mathcal{D}}^{(N)}, \hat{\mathcal{T}}^{(N)}, \hat{\mathbf{X}}_0^{(N)})$ associated with $\mathcal{X}^{(N)} = (\mathbf{X}, \mathcal{D}^{(N)}, \mathcal{T}^{(N)}, \mathbf{X}_0^{(N)})$ is defined by:

- Variables: $\hat{\mathbf{X}} = \frac{\mathbf{X}}{N}$
- Domain: $\hat{\mathcal{D}}^{(N)} = \{N^{-1}\mathbf{x} \mid \mathbf{x} \in \mathcal{D}\}$.
- Initial conditions: $\hat{\mathbf{X}}_n^{(N)} = \frac{\mathbf{X}_0^{(N)}}{N}$
- Normalized transition $\hat{\tau} = (\hat{\phi}_\tau^{(N)}(\hat{\mathbf{X}}), \frac{\mathbf{v}_\tau}{N}, \hat{r}_\tau^{(N)}(\hat{\mathbf{X}}))$ associated with $\tau \in \mathcal{T}^{(N)}$:
  - Update: $\frac{\mathbf{v}_\tau}{N}$;
  - Guards: $\hat{\phi}_\tau^{(N)}(\frac{\mathbf{X}}{N}) = \phi_\tau^{(N)}(\mathbf{X})$
  - Rates: $r_\tau^{(N)}(\mathbf{X}) = Nf_\tau^{(N)}\left(\frac{\mathbf{X}}{N}\right) = \hat{r}_\tau^{(N)}\left(\frac{X}{N}\right)$

EXAMPLE: SIR EPIDEMICS



- $r_{rec}^{(N)}(\mathbf{X}) = k_R X_I = N k_R \frac{X_I}{N} = N k_R \hat{X}_I$
  $\hat{r}_{rec}^{(N)}(\hat{\mathbf{X}}) = N k_R \hat{X}_I$, $f_{rec}(\hat{\mathbf{X}}) = k_R \hat{X}_I$
- $r_{inf}^{(N)}(\mathbf{X}) = \frac{k_I}{N} X_S X_I = N k_I \frac{X_S}{N} \frac{X_I}{N} = N k_I \hat{X}_S \hat{X}_I$
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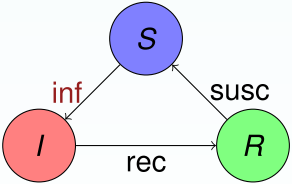

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SCALING ASSUMPTIONS: STATE SPACE

- Consider the normalised state space $\hat{\mathcal{D}}^{(N)}$ of $\hat{\mathbf{X}}^{(N)}(t)$.
- We need to find a set $E \subset \mathbb{R}^n$ (open or compact) which contains $\hat{\mathcal{D}}^{(N)}$ for each N. This will be the set in which the fluid limit will live.

EXAMPLE: SIR EPIDEMICS

In this case, the normalised variables take values in a discrete grid between 0 and 1:
$\hat{\mathcal{D}}_i^{(N)} = \{\frac{j}{N} \mid j = 1, \ldots, N\}$.

Hence, we can take E to be the unit cube $[0, 1]^3$.

However, the total population is conserved, so we can restrict to the unit simplex $E = \{\mathbf{x} \in [0, 1]^3 \mid \sum_i x_i = 1\}$.

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SCALING ASSUMPTIONS

We require that guards are independent of N: $\hat{\phi}_\tau^{(N)}(\mathbf{x}) = \hat{\phi}_\tau(\mathbf{x})$.

$f_\tau^{(N)}$ is required to converge uniformly to a locally Lipschitz continuous and locally bounded function f_τ:

$$\sup_{\mathbf{x}\in E} \|f_\tau^{(N)}(\mathbf{x}) - f_\tau(\mathbf{x})\| \to 0.$$

If $f_\tau^{(N)} = f_\tau$ does not depend on N, the rate satisfies the density dependence condition.

f locally Lipschitz iff $\forall \mathbf{x}, \exists B(\mathbf{x}), L > 0, \forall \mathbf{y} \in B(\mathbf{x})\ \|f(\mathbf{x}) - f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$
f locally bounded iff $\forall \mathbf{x}, \exists B(\mathbf{x}), M > 0, \|f(\mathbf{x})\| \leq M\|\mathbf{x} - \mathbf{y}\|$

The following theorem works also under less restrictive assumptions (e.g. random increments with bounded variance and average).

DRIFT AND LIMIT VECTOR FIELD

DRIFT

The drift or mean increment at level N is

$$F^{(N)}(\mathbf{x}) = \sum_{\tau \in \mathcal{T}} \mathbf{v}_\tau I_{\hat{\phi}_\tau(\mathbf{x})} f_\tau^{(N)}(\mathbf{x})$$

By the scaling assumptions, $F^{(N)}$ converges uniformly to F, the limit vector field:

$$F(\mathbf{x}) = \sum_{\tau \in \mathcal{T}} \mathbf{v}_\tau I_{\hat{\phi}_\tau(\mathbf{x})} f_\tau(\mathbf{x}).$$

FLUID ODE

The fluid ODE is

$$\frac{d\mathbf{x}(t)}{dt} = F(\mathbf{x}(t))$$

DETERMINISTIC APPROXIMATION THEOREM

HYPOTHESIS

- $\hat{\mathbf{X}}^{(N)}(t)$: sequence of Markov processes that satisfy the conditions above.
- F Lipschitz continuous in E.
- $\exists \mathbf{x_0} \in S$ such that $\hat{\mathbf{X}}^{(N)}(0) \to \mathbf{x_0}$ in probability (or almost surely)

$\mathbf{x}(t)$: solution of $\dot{\mathbf{x}} = F(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x_0}$, living in E for all $t \geq 0$.

DETERMINISTIC APPROXIMATION THEOREM

THEOREM (KURTZ)

For any finite time horizon $T < \infty$, it holds that:

$$\sup_{0 \leq t \leq T} \|\hat{\mathbf{X}}^{(N)}(t) - \mathbf{x}(t)\| \to 0 \text{ in probability,}$$

meaning, for each $\delta > 0$, that

$$\lim_{N \to \infty} \mathbb{P}\left\{ \sup_{0 \leq t \leq T} \|\hat{\mathbf{X}}^{(N)}(t) - \mathbf{x}(t)\| > \delta \right\} = 0$$

REMARK

Convergence holds also almost surely:

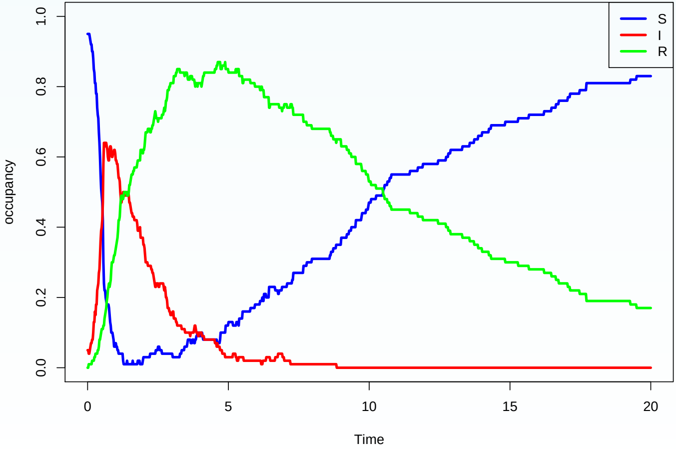
$$\mathbb{P}\left\{ \lim_{N \to \infty} \sup_{0 \leq t \leq T} \|\hat{\mathbf{X}}^{(N)}(t) - \mathbf{x}(t)\| = 0 \right\} = 1$$

EPIDEMICS EXAMPLE CONTINUED

The CTMC $\mathbf{X}^{(N)}(t)$ of the epidemics model satisfies all the hypothesis of fluid limit theorem, so it converges in probability to the solution of the following set of ODEs:

$$\begin{cases} \frac{dx_S}{dt} = k_S x_R - k_I x_I x_S \\ \frac{dx_I}{dt} = k_I x_I x_S - k_R x_I \\ \frac{dx_R}{dt} = k_R x_I - k_S x_R \end{cases}$$

EPIDEMICS EXAMPLE CONTINUED



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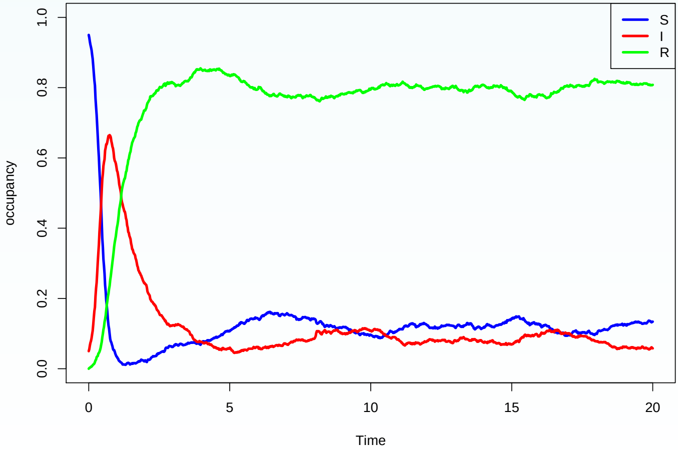

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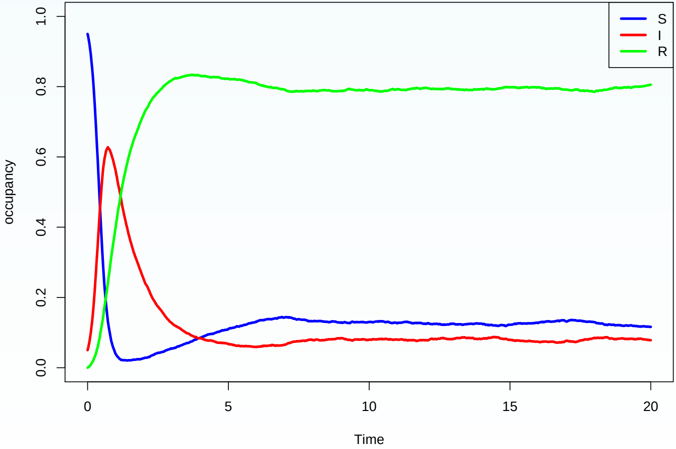

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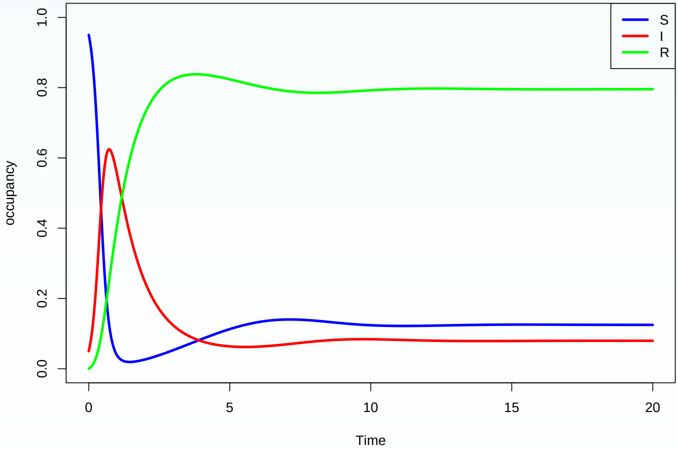

Limit ODE

REMINDER: CONVERGENCE OF RANDOM VARIABLES

ALMOST SURE CONVERGENCE

Let $X, X_1, X_2, \ldots : (\Omega, \mathcal{S}) \to (E, \mathcal{B})$. Then $X_n \to X$ almost surely iff X_n converges to X in a set of probability 1:

$$\mathbb{P}\{\lim_{n\to\infty} \|X_n - X\| = 0\} = 1$$

CONVERGENCE IN PROBABILITY

Let $X, X_1, X_2, \ldots : (\Omega, \mathcal{S}) \to (E, \mathcal{B})$. Then $X_n \to X$ in probability iff for each $\delta > 0$

$$\lim_{n\to\infty} \mathbb{P}\{\|X_n - X\| > \delta\} = 0$$

REMINDER: CONVERGENCE OF RANDOM VARIABLES

CONVERGENCE IN DISTRIBUTION (WEAK CONVERGENCE)

Let $X, X_1, X_2, \ldots$ be random variables with values in $(E, \mathcal{B})$, where E is a Polish space. Then $X_n \Rightarrow X$ (X_n converges weakly to X) iff, for each bounded continuous function $f : E \to \mathbb{R}$, it holds that

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)].$$

WHY CONVERGENCE IN DISTRIBUTION?

Notice that, if $\mu, \mu_1, \mu_2, \ldots$ are the probability distributions in $(E, \mathcal{B})$ associated with $X, X_1, X_2, \ldots$, then the weak convergence of X_n to X is equivalent to $\mu_n \to \mu$ w.r.t the weak topology in the space of probability measures on E.

PROOF OF KURTZ THEOREM: BLACKBOARD!

POISSON REPRESENTATION

$$\hat{\mathbf{X}}^{(N)}(t) = \hat{\mathbf{X}}^{(N)}(0) + \sum_{\eta \in \mathcal{T}} \frac{1}{N} \mathbf{v}_\eta \mathcal{N}_\eta \left(N \int_0^t f_\eta(\hat{\mathbf{X}}^{(N)}(s)) ds \right).$$

ODE SOLUTION, INTEGRAL FORM

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t F(\mathbf{x}(s)) ds$$

GENERAL IDEA: CTMC AS A PERTURBED DYNAMICAL SYSTEM

$$\hat{\mathbf{X}}^{(N)}(t) = \hat{\mathbf{X}}^{(N)}(0) + \int_0^t F(\hat{\mathbf{X}}^{(N)}(s)) ds + D^{(N)}(t),$$

$$D^{(N)}(t) := Xb^{(N)}(t) - \hat{\mathbf{X}}^{(N)}(0) - \int_0^t F(\hat{\mathbf{X}}^{(N)}(s)) ds$$

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PROOF OF KURTZ THEOREM: BLACKBOARD!

CENTERED POISSON PROCESS

Consider a Poisson process $\mathcal{N}\left(\int_0^t \lambda(s)ds\right)$ with time-varying rate $\lambda(t)$. Its centred version is

$$\tilde{\mathcal{N}}\left(\int_0^t \lambda(s)ds\right) = \mathcal{N}\left(\int_0^t \lambda(s)ds\right) - \int_0^t \lambda(s)dsds.$$

LAW OF LARGE NUMBERS FOR CENTERED POISSON PROCESS

Law of large numbers for constant rate: for each $T \geq 0$

$$\sup_{t \leq T} \frac{1}{N} \tilde{\mathcal{N}}(N\lambda t) \to 0 \ \ a.s.$$

GRONWALL'S INEQUALITY

If for $a, b > 0$, $f(t) \leq a + b\int_0^t f(s)ds$, then $f(t) \leq ae^{-bt}$.

KURTZ THEOREM FOR EXIT TIMES

EXIT TIMES

Fix a set $S \subset E$ (the safe set) and suppose we want to estimate the time in which $\hat{\mathbf{X}}^{(N)}(t)$ leaves S. We can use Kurtz theorem for this!

- $S \subseteq E$, open in E. F Lipschitz continuous in S.
- $\zeta(S)$: exit time of $\mathbf{x}(t)$ from S.
- Assume $\mathbf{x}(t)$ leaves S by crossing transversally the boundary ∂S.
- $\zeta^{(N)}(S)$: exit time of $\hat{\mathbf{X}}^{(N)}(t)$ from S.

THEOREM (KURTZ FOR EXIT TIMES)

If $\zeta(S) < \infty$, it holds that:

$$\|\zeta^{(N)}(S) - \zeta(S)\| \to 0 \text{ in probability (a.s.).}$$

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Outline

DEFINITION OF SEMIGROUP AND INFINITESIMAL GENERATORS

SEMIGROUP H_t OF A STOCHASTIC PROCESS $\mathbf{X}(t)$ ON E

Let $\mathcal{C}_0(E)$ be the space of continuous functions on E vanishing at infinity, and let $f \in \mathcal{C}_0(E)$.

$$H_t f(\mathbf{x}) = \mathbb{E}[f(\mathbf{X}(t)) \mid \mathbf{X}(0) = \mathbf{x}]$$

INFINITESIMAL GENERATOR A OF A STOCHASTIC PROCESS $X(t)$

It is an operator $A : \mathcal{D}(A) \subseteq \mathcal{C}_0(E) \to \mathcal{C}_0(E)$ defined by

$$Af = \lim_{t \to 0^+} \frac{1}{t}(H_t f - f) \quad \text{uniformly.}$$

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LIMIT THEOREM FOR INFINITESIMAL GENERATORS

THEOREM (EITHER AND KURTZ, 1986)

Let $X, X^{(1)}, X^{(2)}, \ldots$ be Feller processes in the state space E, with semigroups $H_t, H_t^{(1)}, H_t^{(2)}, \ldots$ and generators $A, A_1, A_2, \ldots$. Let D be a core for A. The following statements are equivalent:

1. if $f \in D$, there exists some f_n in $\mathcal{D}_{A_n}$ with $f_n \to f$ and $A_n f_n \to Af$;
2. $H_t^{(n)} f \to H_t f$ for each $f \in \mathcal{C}_0(S)$ and $t > 0$;
3. $H_t^{(n)} f \to H_t f$ for each $f \in \mathcal{C}_0(S)$, uniformly on bounded intervals.
4. if $X_0^{(n)} \Rightarrow X_0$, then $X^{(n)} \Rightarrow X$ (convergence in distribution).

INFINITESIMAL GENERATORS OF CTMC AND ODE

INFINITESIMAL GENERATOR OF THE CTMC

Consider a population CTMC $\mathcal{X} = (\mathbf{X}, \mathcal{D}, \mathcal{T}, \mathbf{X}_0)$, then its infinitesimal generator is

$$Af(\mathbf{x}) = \sum_{\eta \in \mathcal{T}} r_\eta(\mathbf{x})(f(\mathbf{x} + \mathbf{v}_\eta) - f(\mathbf{x}))$$

For a CTMC specified by a Q-matrix, the infinitesimal generator is $\mathcal{A}\mathbf{f} = Q\mathbf{f}$ ($\mathbf{f}$ is a vector if S is countable).

INFINITESIMAL GENERATOR OF AN ODE

Consider a vector field $F : E \to \mathbb{R}^n$ and the associated ODE $\frac{d\mathbf{x}(t)}{dt} = F(\mathbf{x}(t))$.

A is the directional derivative of f along the flow of F. For $f \in C_0^1(E)$:

$$Af(\mathbf{x}) = \langle \nabla f(\mathbf{x}), F(\mathbf{x}) \rangle$$

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ONE SLIDE PROOF OF KURTZ THEOREM!

We just need to prove that $A_N g \to Ag$ for each $g \in \mathcal{C}_0^1(E)$, where A_N is the generator of a sequence $\hat{X}^{(N)}$ of normalized population CTMC and A is the generator of the limit ODE.

$$\begin{aligned}
A_N g(\mathbf{x}) &= \sum_{\eta\in\mathcal{T}} \hat{r}_\eta^{(N)}(\mathbf{x})\left(g(\mathbf{x}+\frac{1}{N}\mathbf{v}_\eta)-g(\mathbf{x})\right)\\
&= \sum_{\eta\in\mathcal{T}} f_\eta^{(N)}(\mathbf{x})\frac{(g(\mathbf{x}+\frac{1}{N}\mathbf{v}_\eta)-g(\mathbf{x}))}{\frac{1}{N}}\\
&\to \sum_{\eta\in\mathcal{T}} f_\eta(\mathbf{x})\langle\nabla g(\mathbf{x}),\mathbf{v}_\eta\rangle\\
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where F is the fluid limit of the sequence of CTMC:

$$F(\mathbf{x}) = \sum_{\eta\in\mathcal{T}} \mathbf{v}_\eta f_\eta(\mathbf{x})$$

ONE SLIDE PROOF OF KURTZ THEOREM!

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Outline

Stationary Regime

- The fluid approximation and mean field theorems provide conditions for the convergence up to any finite time horizon.
- They do not predict convergence of the stationary regime.
- This is because they hold for any possible trajectory of the ODE, including unstable ones.
- In order to provide some result for the stationary behaviour, one has to look at the Phase Space Properties of the system of ODEs.

SOME DEFINITIONS

DEFINITIONS

- Flow of the ODE: $\xi(t, x)$
- Orbit of the flow, starting from x: $\gamma(x)$
- Forward orbit of the flow, starting from x: $\gamma^+(x)$
- Invariant set A iff $\gamma(x) \subset A$, for $x \in A$
- Attractor: invariant set A such that there is a neighborhood U of A with $\lim_{t\to\infty} d_H(\xi(t, x), A) = 0$ uniformly for $x \in U$
- Basin of attraction of A: $B(A) = \{x \in E \mid \lim_{t\to\infty} d_H(\xi(t, x), A) = 0\}$

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BIRKHOFF CENTRE AND INVARIANT MEASURES

BIRKHOFF CENTRE OF A FLOW

The Birkhoff centre $B(\xi)$ of a flow ξ is, informally, the set of limit points of the flow (steady states, limit circles, etc.).

INVARIANT MEASURE OF A FLOW

A probability measure μ on $(E\,\mathcal{B})$ is invariant for the flow ξ iff for each $A \in \mathcal{B}$ and $t \geq 0$

$$\mu(\xi^{-1}(t, A)) = \mu(A).$$

INVARIANT MEASURES AND BIRKHOFF CENTRE

Any invariant probability measure μ for the flow ξ has support contained in $B(\xi)$.

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Convergence of Invariant Measures

Theorem

Let $\mu^{(N)}$ be an invariant measure for $\mathbf{X}^{(N)}(t)$. Any limit point μ (w.r.t. the weak topology) of the sequence $\mu^{(N)}$ is an invariant measure of the flow ξ.

In other words: $\mathbf{X}^{(N)}(t)$ spends most of its time close to the Birkhoff centre $B(\xi)$ of the flow.

Corollary

If $X^{(N)}(t)$ are irreducible and the ODE have a unique globally attracting stable fixed point $\bar{x}$, then $\mu^{(N)} \to \mu$, where μ concentrates the mass on $\bar{x}$.

Fixed Point Method

The fixed point method for mean field analysis approximates the stationary distribution with the value of the occupancy measure of the ODE fixed, if it is unique.
However, global attractiveness has to be proved.

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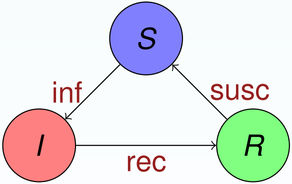
EXAMPLE: SIR EPIDEMICS

Global attractiveness is a crucial property. Consider again the SIR model and the set of fluid equations.



$$\begin{cases} \frac{dx_S}{dt} = k_S x_R - k_I x_I x_S \\ \frac{dx_I}{dt} = k_I x_I x_S - k_R x_I \\ \frac{dx_R}{dt} = k_R x_I - k_S x_R \end{cases}$$

ODEs have two fixed points: $\left(\frac{k_R}{k_I}, \frac{k_S(k_I - k_R)}{k_I(k_S + k_R)}, \frac{k_R(k_I - k_R)}{k_I(k_S + k_R)}\right)$, if $\frac{k_R}{k_I} < 1$, and $(1, 0, 0)$

No matter how large is N, all trajectories of the CTMC will eventually reach the state in which the epidemics is extinct: the steady state measure of $\hat{\mathbf{X}}^{(N)}$ is the Dirac delta on $(1, 0, 0)$.

Outline

REWARDS

Reward measures are a very useful companion of CTMC population models. They allow us to capture useful measures, like the throughput of a system, or the energy consumption.

We consider here two classes of reward measures, all state-based.

REWARD FUNCTION

$\rho : E \to \mathbb{R}_{\geq 0}$ is the reward associated to a state $\mathbf{x} \in E$.
We assume ρ is continuous in E.

We assume rewards depend on the normalised state.

INSTANTANEOUS AND CUMULATIVE REWARDS

INSTANTANEOUS REWARD

The expected value of ρ at time t

$$R_I^{(N)}(t) = \mathbb{E}\left[\rho(\hat{\mathbf{X}}^{(N)}(t))\right]$$

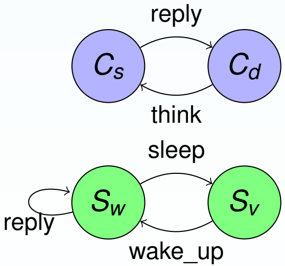
CUMULATIVE REWARDS

The expected reward accumulated up to time t

$$R_C^{(N)}(t) = \mathbb{E}\left[\int_0^t \rho(\hat{\mathbf{X}}^{(N)}(s))ds\right]$$

EXAMPLE: QUEUE MODEL WITH SERVER VACATION

Consider a model of a closed queue network, with a (M/M/m) service station and a delay station, and assume servers can take a vacation, to save energy.



Four variables: C_s, C_d, S_w, S_v.

- $(\mathit{reply}, \top, (-1, +1, 0, 0), k_r \min\{C_s, S_w\})$
- $(\mathit{think}, \top, (+1, -1, 0, 0), k_t C_d)$
- $(\mathit{sleep}, \top, (0, 0, -1, +1), k_v S_w)$
- $(\mathit{wake\_up}, \top, (0, 0, +1, -1), k_w S_v)$

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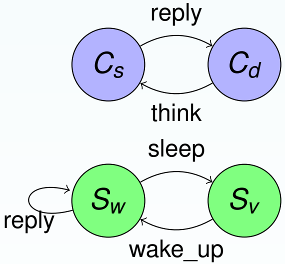

Four variables: C_s, C_d, S_w, S_v.
αN clients, βN servers

- $(reply, \top, (-1, +1, 0, 0), k_r \min\{C_s, S_w\})$
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$$(c_s, c_d, c_w, c_v) = (C_s, C_d, S_w, S_v)/N$$

REWARDS

THROUGHPUT: $\rho_t(c_s, c_d, c_w, c_v) = k_r \min\{c_s, s_w\})$

ENERGY CONSUMPTION: $\rho_u(c_s, c_d, c_w, c_v) = u_s \cdot s_w$

Convergence of Instantaneous Rewards

$\rho : E \to \mathbb{R}$ is a (bounded) continuous function

Continuous Mapping Theorem

If $\mathbf{X}^{(N)} \to \mathbf{X}$ (a.s./ in prob.) and f is $\mathbf{X}$-a.s. continuous (i.e. $f(\mathbf{X})$ is continuous with probability one), then $f(\mathbf{X}^{(N)}) \to f(\mathbf{X})$.

Bounded convergence

If $\mathbf{X}^{(N)} \to \mathbf{X}$ (a.s./ in prob.) and $\mathbb{E}[\mathbf{X}] < \infty$ and $\|\mathbf{X}^{(N)}\| \leq M$ for each N, then $\mathbb{E}[\|\mathbf{X}^{(N)} - \mathbf{X}\|] \to 0$ (convergence in mean).

Corollary (of Kurtz theorem)

$$\mathbb{E}[\rho(\hat{\mathbf{X}}^{(N)}(t))] \to \mathbb{E}[\rho(\mathbf{x}(t))] = \rho(\mathbf{x}(t))$$

CONVERGENCE OF CUMULATIVE REWARDS

$f_C(\mathbf{x}) = \int_0^t \rho(\mathbf{x}(s))ds$, for fixed t, can be seen as a functionals of a trajectory $\mathbf{x}$ of the CTMC, which is a cadlag function with values in E. Call $\mathcal{E}$ this set, then $f_C : \mathcal{E} \to \mathbb{R}$

WEAK CONVERGENCE

Let $\mathbf{X}^{(N)}, \mathbf{X}$ have values in $\mathcal{E}$. $\mathbf{X}^{(N)} \Rightarrow \mathbf{X}$ (weakly) if and only if, for each continuous and bounded functional $f : \mathcal{E} \to \mathbb{R}$, it holds that

$$\mathbb{E}[f(\mathbf{X}^{(N)})] \to \mathbb{E}[f(\mathbf{X})]$$

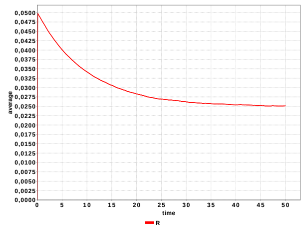
COROLLARY (OF KURTZ THEOREM)

By Kurtz theorem $\hat{\mathbf{X}}^{(N)} \Rightarrow \mathbf{x}$ (weakly), and (if E is compact) f_C is a continuous and bounded functional, so that:

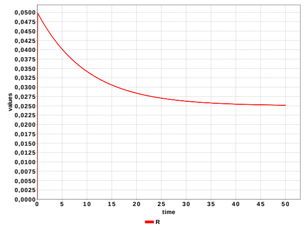
$$\mathbb{E}\left[\int_0^t \rho(\hat{\mathbf{X}}^{(N)}(s))ds\right] \to \mathbb{E}\left[\int_0^t \rho(\mathbf{x}(s))ds\right] = \int_0^t \rho(\mathbf{x}(s))ds$$

EXAMPLE: QUEUE MODEL WITH SERVER VACATION

Throughput

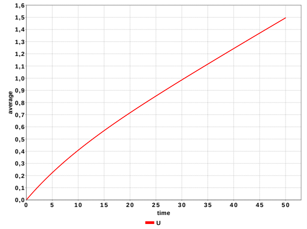


$N = 1000$



fluid

Energy consumption



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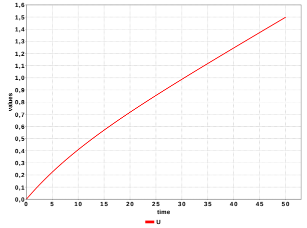

fluid

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